Derivation of a compressible bubbly flow model

École du GdR MathGéoPhy
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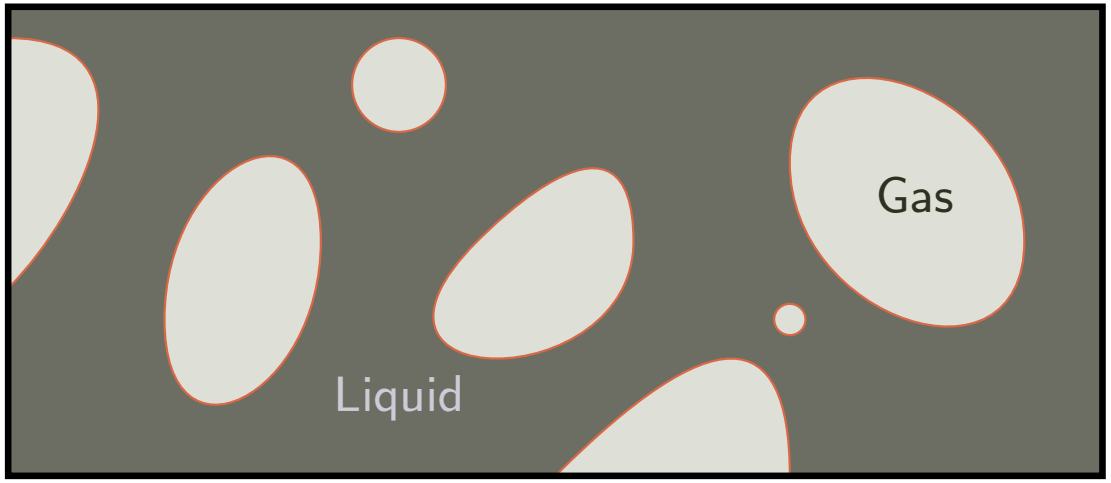
Context

- Compressible multiphase flows with heterogeneities (bubbles, droplets)



- Average model, macroscopic description
- Rigorous derivation
  - Full model: PDEs and source terms
  - Mathematical theory

Derivation of averaged models

- **Averaging approach**
  [Drew & Passman '98, Ishii & Hibiki '06,...]
  - Microscopic description
    - Instantaneous local conservation laws for each separated phase
    - Jump conditions through the interfaces
  - Averaging process
    - Introduce time and/or volume scales, or random disturbances
    - Average the microscopic model wrt the small scales
  - ✓ Baer-Nunziato type model
    - 2 Euler systems, nonconservative coupling terms
    - Transport of a void fraction, mechanical relaxation source terms
    - Kinetic and thermodynamical relaxation source terms
  - ✗ Closure laws
  - ✗ Definition of the averaging operators

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- **Homogenization approach**
  [Serre '91 & '01, E '92, Hillairet '07, Bresch & Huang '11, Bresch, Hillairet '15 & '19, Hillairet '18, Bresch, Burtea & Lagoutière '20,...]

Homogenization for two-phase flows

Standard approach

- Microscopic description
  - Viscous flows (smooth enough solutions) for both phase
  - Conditions through the interfaces ("perfect transducers")
- One-fluid model with high-oscillatory density solutions
- Pass to the limit to deduce macroscopic quantities $\bar{\alpha}_{f,g}, \bar{\rho}_{f,g}, \bar{\rho}$ and $\bar{u}$

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$$\begin{cases} \bar{\alpha}_f + \bar{\alpha}_g = 1, \qquad \bar{\rho} = \bar{\alpha}_f \bar{\rho}_f + \bar{\alpha}_g \bar{\rho}_g \\ \partial_t \bar{\alpha}_f + \bar{u} \partial_x \bar{\alpha}_f = \dfrac{\bar{\alpha}_g \bar{\alpha}_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f} \Big[(\mu_g - \mu_f) \partial_x \bar{u} + (\mathrm{p}_f(\bar{\rho}_f) - \mathrm{p}_g(\bar{\rho}_g)) \Big] \\ \partial_t (\bar{\alpha}_f \bar{\rho}_f) + \partial_x (\bar{\alpha}_f \bar{\rho}_f \bar{u}) = 0, \qquad \partial_t (\bar{\alpha}_g \bar{\rho}_g) + \partial_x (\bar{\alpha}_g \bar{\rho}_g \bar{u}) = 0 \\ \partial_t (\bar{\rho} \bar{u}) + \partial_x (\bar{\rho} \bar{u}^2) = \partial_x \bar{\Sigma} \\ \text{with } \bar{\Sigma} = \frac{\mu_g \mu_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f} \left[\partial_x \bar{u} - \left(\frac{\bar{\alpha}_f}{\mu_f} \mathrm{p}_f(\bar{\rho}_f) + \frac{\bar{\alpha}_g}{\mu_g} \mathrm{p}_g(\bar{\rho}_g) \right) \right] \end{cases}$$

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Pros & Cons

✓ Fully rigorous

✗ Simple interface behaviors, one-velocity models

- Extensions: different EoS [Bresch & Hillairet '19], temperature [Hillairet '21], density overlap [Bresch, Burtea & Lagoutière '20]

Homogenization for two-phase flows

Standard approach

- Microscopic description
  - Viscous flows (smooth enough solutions) for both phase
  - Conditions through the interfaces ("perfect transducers") ✗
- One-fluid model with high-oscillatory density solutions ✗
- Pass to the limit to deduce macroscopic quantities $\bar{\alpha}_{f,g}, \bar{\rho}_{f,g}, \bar{\rho}$ and $\bar{u}$

$$\begin{cases} \bar{\alpha}_f + \bar{\alpha}_g = 1, \qquad \bar{\rho} = \bar{\alpha}_f \bar{\rho}_f + \bar{\alpha}_g \bar{\rho}_g \\ \partial_t \bar{\alpha}_f + \bar{u} \partial_x \bar{\alpha}_f = \dfrac{\bar{\alpha}_g \bar{\alpha}_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f} \left[(\mu_g - \mu_f) \partial_x \bar{u} + (\mathrm{p}_f(\bar{\rho}_f) - \mathrm{p}_g(\bar{\rho}_g)) + \dots \right] \\ \partial_t (\bar{\alpha}_f \bar{\rho}_f) + \partial_x (\bar{\alpha}_f \bar{\rho}_f \bar{u}) = 0, \qquad \partial_t (\bar{\alpha}_g \bar{\rho}_g) + \partial_x (\bar{\alpha}_g \bar{\rho}_g \bar{u}) = 0 \\ \partial_t (\bar{\rho} \bar{u}) + \partial_x (\bar{\rho} \bar{u}^2) = \partial_x \bar{\Sigma} \\ \text{with } \bar{\Sigma} = \frac{\mu_g \mu_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f} \left[\partial_x \bar{u} - \left(\frac{\bar{\alpha}_f}{\mu_f} \mathrm{p}_f(\bar{\rho}_f) + \frac{\bar{\alpha}_g}{\mu_g} \mathrm{p}_g(\bar{\rho}_g) \right) + \dots \right] \\ + \text{Additional relations} \end{cases}$$

Goal [Hillairet, M. & Seguin '22]

- Introduce more complex interface behavior
- Couple different fluid models

Outline/Methodology

$$\begin{tikzcd}
\mathrm{IC}^{(N)} \arrow[r] & \mathcal{M}^{(N)} \arrow[d, "4"] \\
\overline{\mathrm{IC}} \arrow[u, "2"] & \overline{\mathcal{M}}
\end{tikzcd}$$

1 3 5

1. Microscopic model with N bubbles
2. Macroscopic to microscopic initial data
3. Solve the microscopic model
4. Pass to the limit $N \to \infty$ to deduce macroscopic quantities
5. Find the associated macroscopic evolution equations

The microscopic model with N bubbles

Compressible Navier-Stokes equations

$$\begin{cases} \partial_t \rho_i + \mathrm{div}(\rho_i u_i) = 0 \\ \partial_t(\rho_i u_i) + \mathrm{div}(\rho_i u_i \otimes u_i) = \mathrm{div}\Sigma_i \\ \text{with } \Sigma_i = 2\lambda_i\big(D(u_i) - \frac{1}{3}\mathrm{div}u_i \mathbb{I}_3\big) + \mu_i \mathrm{div}u_i - \mathrm{p}_i(\rho_i)\mathbb{I}_3 \end{cases}$$

where $i = f$ (the fluid) or $i = k$ (the k-th inclusion B_k), with $k = 1, \ldots, N$

The microscopic model with N bubbles

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Jump conditions

On each bubble boundary ∂B_k:

- Continuity of the velocity field $\quad u_f = u_k$
- Surface tension $\quad (\Sigma_f - \Sigma_k)n = \gamma_s n$

The microscopic model with N bubbles

Compressible Navier-Stokes equations

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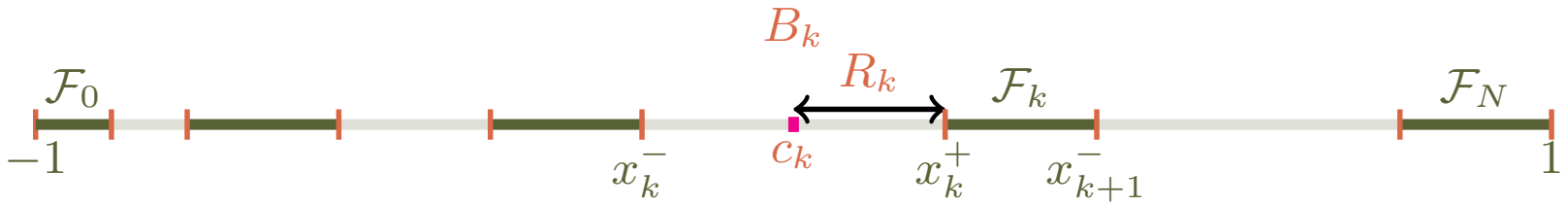
Geometrical assumption

The bubbles B_k remain spherical (translation, rotation, dilatation)

$$\Longrightarrow D(u_k) - \frac{1}{3}\mathrm{div}u_k \mathbb{I}_3 = 0$$

The one-dimensional microscopic model



On the fluid domain $\mathcal{F}(t)$:

$$\begin{cases} \partial_t \rho_f + \partial_x(\rho_f u_f) = 0 \\ \partial_t(\rho_f u_f) + \partial_x(\rho_f (u_f)^2) = \partial_x \Sigma_f \\ \text{with } \Sigma_f = \mu_f \partial_x u_f - \mathrm{p}_f(\rho_f) \end{cases}$$

In each bubble $B_k(t) = B(c_k(t), R_k(t)) = (x_k^-, x_k^+)$ of (constant) mass m_k:

$$\begin{cases} \rho_k(t) = \dfrac{m_k}{2R_k(t)} \\ u_k(t,x) = \dot{c}_k + \dfrac{\dot{R}_k}{R_k}(x - c_k) \end{cases} \qquad \begin{cases} m_k \ddot{c}_k = \Sigma_f(t, x_k^+) - \Sigma_f(t, x_k^-) \\ \dfrac{m_k}{3}\ddot{R}_k = \Sigma_f(t, x_k^-) + \Sigma_f(t, x_k^+) - 2\Sigma_k + \dfrac{\gamma_s}{R_k} \\ \text{with } \Sigma_k = \mu_g \partial_x u_k - \mathrm{p}_g\left(\dfrac{m_k}{2R_k}\right) \end{cases}$$

✓ Well-posedness of the Cauchy problem (up to assumptions on the initial data)

Outline/Methodology

1. Microscopic model with N bubbles
2. Macroscopic to microscopic initial data
3. Solve the microscopic model
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Macroscopic to microscopic initial data

At the macroscopic scale, both fluids are present everywhere in the domain Ω

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Macroscopic initial data

- Density of the fluid $\bar{\rho}_f^0 \geq \rho_{\text{min}} \in H^1(\Omega)$
- Density of the gas $\bar{\rho}_g^0 \geq \rho_{\text{min}} \in H^1(\Omega)$
- Mean velocity $\bar{u}^0 \in H^1(\Omega)$
- ✗ Void fraction $\bar{\alpha}_g^0$

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- ✗ Void fraction $\bar{\alpha}_g^0$

Reconstruction of gas-bubble distribution

- Probability distribution of the bubbles, in position x and radius r

$$\bar{S}_g^0 = \bar{S}_g^0(x,r) \in L^1(\Omega \times \mathbb{R}^+)$$

Moments of the probability distribution $\bar{S}_g^0$:

- $1-$st order moment $\rightsquigarrow$ void fraction

$$\bar{\alpha}_g^0(x) = \int_{\mathbb{R}^+} (2r)\bar{S}_g^0(x,r)\mathrm{d}r$$

- $0-$th order moment ($\rightsquigarrow$ gas "interfacial area")

$$\bar{f}_g^0(x) = \int_{\mathbb{R}^+} \bar{S}_g^0(x,r)\mathrm{d}r$$

Macroscopic to microscopic initial data

Family of microscopic initial data to be constructed from $\bar{S}_g^0$, $\bar{\rho}_{f,g}^0$ and $\bar{u}^0$

For any bubble number $N \geq 1$:

1. Define a bubble distribution from $\bar{S}_g^0$ to get $(c_k^{(N)}, R_k^{(N)})_{k=1,\dots,N}$
2. Define the densities
$$\begin{cases} \rho_f^{(N)}(0,x) & \text{on } \mathcal{F}^{(N)}(0) \\ \rho_k^{(N)}(0,x) & \text{on } B_k^{(N)}(0) \end{cases}$$
3. Define the velocities
$$\begin{cases} u_f^{(N)}(0,x) & \text{on } \mathcal{F}^{(N)}(0) \\ \dot{c}_k^{(N),0} \text{ and } \dot{R}_k^{(N),0} & \text{on } B_k^{(N)}(0) \end{cases}$$

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Solve the microscopic model

Scaling

- m_k, R_k^0, $|\mathcal{F}_k^0|$ and γ_s behave as N^{-1}

Microscopic Cauchy problem, independent of N [Hillairet, M., Seguin '22]

Consider compatible initial data and the scaling. Then there exists $T_\infty > 0$, independent of N, such that

$$\left((c_k^{(N)}, R_k^{(N)})_{k=1,\dots,N}, \rho_f^{(N)}, u_f^{(N)}, (\rho_k^{(N)}, u_k^{(N)})_{k=1,\dots,N}\right)$$

exists and is unique

- T_∞ taken smaller and smaller along the proof
- Combine energy and regularity estimates, independent of N
- Smoothness of the velocity $u_f^{(N)}$ obtained by extended stress tensors

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Convergence results

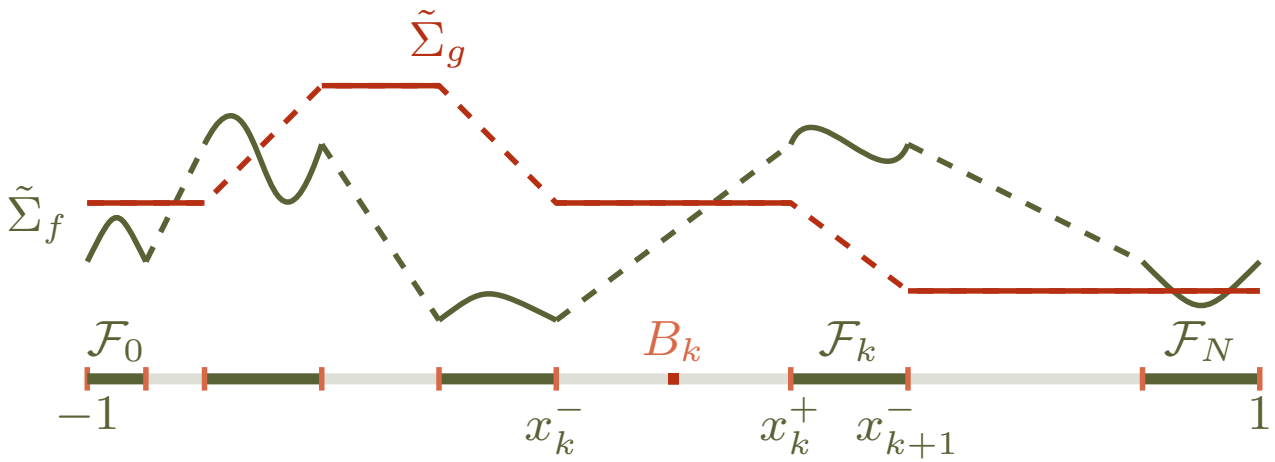
- Due to relative compactness (up to extraction of subsequences)

Mixture unknowns

- Linear extended velocity

$$\tilde{u}^{(N)} = u_f^{(N)} \text{on } \mathcal{F}^{(N)} \text{and } u_k^{(N)} \text{on } B_k^{(N)},\ k = 1, \dots, N$$

  such that $\tilde{u}^{(N)} \to \bar{u}$ in $L^2((0,T); L^2(\Omega))$ when $N \to +\infty$
- Linear extension of Σ_f and Σ_g over $\Omega \rightsquigarrow$ Distinct stress tensors



  such that

$$\tilde{\Sigma}_f^{(N)}, \tilde{\Sigma}_g^{(N)} \rightharpoonup \bar{\Sigma}_f, \bar{\Sigma}_g \text{ in } L^2((0,T); H^1(\Omega)) \text{ when } N \to +\infty$$

Convergence results

- Evolution equations on fluid and bubble unknowns

Bubble unknowns

- Density $\rho_g^{(N)} = \sum_{k=1}^N \rho_k^{(N)} \mathbf{1}_{B_k}$ with $\rho_k^{(N)} = \dfrac{m_k^{(N)}}{2R_k}$
- Interfacial area/covolume $f_g^{(N)} = \sum_{k=1}^N f_k^{(N)} \mathbf{1}_{B_k}$ with $f_k^{(N)} = \dfrac{1}{2NR_k}$

$$\rho_g^{(N)}, f_g^{(N)} \longrightarrow \bar{\rho}_g, \bar{f}_g \text{ in } L^2((0,T_0)\times\Omega) \text{ when } N \to +\infty$$

Fluid unknowns

- Characteristic function of the fluid domain $\chi^{(N)} = \mathbf{1}_{\mathcal{F}^{(N)}}$

$$\chi^{(N)} \rightharpoonup \bar{\alpha}_f \text{ in } L^\infty((0,T)\times\Omega) - w^* \quad \text{and} \quad 0 \le \bar{\alpha}_f \le 1 \ a.e$$

- Density (defined on Ω)

$$\tilde{\rho}_f^{(N)} \longrightarrow \bar{\rho}_f \quad \text{in } L^2((0,T_0)\times\Omega) \quad \text{when} \quad N \to +\infty$$

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Which evolution equations do we expect?

- Immiscibility constraint
- Void fraction equation $\bar{\alpha}_{f,g}$, accounting for mechanical relaxation
- Partial mass conservations with $\bar{\alpha}_f \bar{\rho}_f$ and $\bar{\alpha}_g \bar{\rho}_g$
- Momentum equation with $\bar{\rho}\bar{u}$ with mixture density $\bar{\rho}$

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$\rightsquigarrow$ Pass to the limit in nonlinear combinations of $\chi^{(N)}(t,x)$, $\rho^{(N)}(t,x)$ and $f_g^{(N)}(t,x)$

✓ Nonlinear convergence (in the sense of Young measures)

The key ingredient

Nonlinear function $b \in C^1([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+)$ and consider the sequence

$$b^{(N)}(t,x) = b(\chi^{(N)}(t,x), \rho^{(N)}(t,x), f_g^{(N)}(t,x)) \quad \forall (t,x) \in (0,T) \times \Omega$$

Nonlinear convergence

The sequence $(b^{(N)})$ satisfies

$$\partial_t b^{(N)} + \partial_x (b^{(N)} \tilde{u}^{(N)}) + \big(\partial_2 b^{(N)} \rho^{(N)} + \partial_3 b^{(N)} f_g^{(N)} - b^{(N)}\big) \partial_x \tilde{u}^{(N)} = 0 \quad \text{in } \mathcal{D}'((0,T) \times \Omega)$$

Moreover, there exists $\bar{b} \in L^\infty((0,T) \times \Omega)$ such that

$$b^{(N)} \rightharpoonup \bar{b}, \quad \text{in } L^\infty((0,T) \times \Omega) - w^\star \text{ when } N \to +\infty$$

Some examples

Considering $b^{(N)} = \chi^{(N)}$ and $b^{(N)} = 1 - \chi^{(N)}$ gives $\bar{b} = \bar{\alpha}_f$ and $\bar{b} = \bar{\alpha}_g$

The immiscibility constraint holds

$$\bar{\alpha}_f + \bar{\alpha}_g = 1$$

Some examples

Considering $b^{(N)} = \chi^{(N)}$ and $b^{(N)} = 1 - \chi^{(N)}$ gives $\bar{b} = \bar{\alpha}_f$ and $\bar{b} = \bar{\alpha}_g$

The immiscibility constraint holds

$$\bar{\alpha}_f + \bar{\alpha}_g = 1$$

Considering $b^{(N)} = f_g^{(N)}$ gives $\bar{b} = \bar{f}_g$

The interfacial area $\bar{f}_g$ satisfies

$$\partial_t \bar{f}_g + \partial_x (\bar{f}_g \bar{u}) = 0$$

The macroscopic model

Macroscopic closed system

$$
\begin{cases}
\bar{\alpha}_f + \bar{\alpha}_g = 1, \qquad \bar{\rho} = \bar{\alpha}_f \bar{\rho}_f + \bar{\alpha}_g \bar{\rho}_g \\
\partial_t \bar{\alpha}_f + \bar{u} \partial_x \bar{\alpha}_f = \dfrac{\bar{\alpha}_g \bar{\alpha}_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f} \left[(\mu_g - \mu_f) \partial_x \bar{u} + (\mathrm{p}_f(\bar{\rho}_f) - \mathrm{p}_g(\bar{\rho}_g)) - \bar{\gamma}_s \dfrac{\bar{f}_g}{\bar{\alpha}_g} \right] \\
\partial_t \bar{f}_g + \partial_x (\bar{f}_g \bar{u}) = 0 \\
\partial_t (\bar{\alpha}_f \bar{\rho}_f) + \partial_x (\bar{\alpha}_f \bar{\rho}_f \bar{u}) = 0, \qquad \partial_t (\bar{\alpha}_g \bar{\rho}_g) + \partial_x (\bar{\alpha}_g \bar{\rho}_g \bar{u}) = 0 \\
\partial_t (\bar{\rho} \bar{u}) + \partial_t (\bar{\rho}^2 \bar{u}) = \partial_x \bar{\Sigma} \\
\text{with } \bar{\Sigma} = \frac{\mu_g \mu_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f} \left[\partial_x \bar{u} - \left(\frac{\bar{\alpha}_f}{\mu_f} \mathrm{p}_f(\bar{\rho}_f) + \frac{\bar{\alpha}_g}{\mu_g} \mathrm{p}_g(\bar{\rho}_g) \right) - \frac{\bar{\gamma}_s}{\mu_g} \bar{f}_g \right]
\end{cases}
$$

The macroscopic model

Macroscopic closed system

$$\begin{cases} \bar\alpha_f + \bar\alpha_g = 1, \qquad \bar\rho = \bar\alpha_f \bar\rho_f + \bar\alpha_g \bar\rho_g \\ \partial_t \bar\alpha_f + \bar u \partial_x \bar\alpha_f = \dfrac{\bar\alpha_g \bar\alpha_f}{\bar\alpha_f \mu_g + \bar\alpha_g \mu_f} \left[(\mu_g - \mu_f)\partial_x \bar u + (\mathrm{p}_f(\bar\rho_f) - \mathrm{p}_g(\bar\rho_g)) - \bar\gamma_s \dfrac{\bar f_g}{\bar\alpha_g} \right] \\ \partial_t \bar f_g + \partial_x(\bar f_g \bar u) = 0 \\ \partial_t(\bar\alpha_f \bar\rho_f) + \partial_x(\bar\alpha_f \bar\rho_f \bar u) = 0, \qquad \partial_t(\bar\alpha_g \bar\rho_g) + \partial_x(\bar\alpha_g \bar\rho_g \bar u) = 0 \\ \partial_t(\bar\rho \bar u) + \partial_t(\bar\rho^2 \bar u) = \partial_x \bar\Sigma \\ \text{with } \bar\Sigma = \frac{\mu_g \mu_f}{\bar\alpha_f \mu_g + \bar\alpha_g \mu_f} \left[\partial_x \bar u - \left(\frac{\bar\alpha_f}{\mu_f} \mathrm{p}_f(\bar\rho_f) + \frac{\bar\alpha_g}{\mu_g} \mathrm{p}_g(\bar\rho_g) \right) - \frac{\bar\gamma_s}{\mu_g} \bar f_g \right] \end{cases}$$

Additional kinetic equation

- Distribution function in position and radius $S_t^{(N)} = \dfrac{1}{N} \sum_{k=1}^N \delta_{c_k(t), N R_k(t)}$

$$\langle S_t^{(N)}, \beta \rangle \to \langle \bar S_g, \beta \rangle, \quad \text{in } C([0,T])$$

- Probability distribution $\bar S_g$ satisfies

$$\partial_t \bar S_g - \partial_x(\bar S_g \bar u) + \frac{1}{\mu_g} \partial_r \big((r(\bar\Sigma_g + \mathrm{p}_g(\bar\rho_g)) + \bar\gamma_s/2) \bar S_g \big) = 0$$

- 0-th order moment $\bar f_g(x) = \int_{\mathbb{R}^+} \bar S_g(x,r) \mathrm{d}r$

To sum up

Comments on the macroscopic model

- Two-pressure one-velocity two-phase flow model
  - Both phases are compressible and viscous
  - Extension of Bresch & Hillairet models: mechanical relaxation, surface tension, not a "one-fluid" model
- Additional description
  - New variable $\bar{f}_g$: interfacial area in 3D?
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Thank you for your attention!