Dispersive-hypebolic models for breaking waves

Maria Kazakova
Laboratoire de Mathématiques, Université Savoie Mont Blanc

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École MathGeoPhy
IHP, Paris

Projet: Dispersive-hypebolic models for breaking waves and sediment transport

Collaborateurs: Gaël Richard, Julien Chauchat
Frederic Couderc, Rémy Baraille, Jean-Paul Vila, Arnaud Duran
Yen-Chung Hung

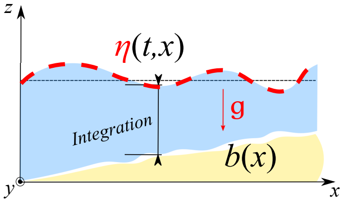
Water waves models

Free-surface incompressible Euler

$t > 0, \vec{x} \in (\mathbb{R}^3, b(\vec{x}) < z < \boldsymbol{\eta}(t, \vec{x}))$

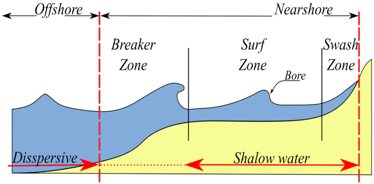
$$\begin{cases} u_t + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + \boldsymbol{g} \\ \nabla \cdot u = 0, \quad \boldsymbol{g} = (0, 0, -g) \end{cases}$$

$+$ kinematic and dynamic boundary conditions



Water waves models

$\mu = H^2/L^2$ (shallowness)
$\varepsilon = a/H$ (nonlinearity)



Hydrostatic pressure

constant velocity over vertical

$u(t,x,z) = \boldsymbol{v}(t,x)$

$$\begin{cases} \dfrac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{v}) = 0, & \text{(Mass Eq)} \\ \dfrac{\partial h\boldsymbol{v}}{\partial t} + \nabla \cdot \left(h\boldsymbol{v} \otimes \boldsymbol{v} + \dfrac{gh^2}{2}\mathcal{I} + p_{NH} \right) = 0, & \text{(Momentum Eq).} \end{cases}$$

model	**NSWE** $\mathcal{O}(\mu)$	$\mathcal{O}(\varepsilon\mu)$	**SGN** $\mathcal{O}(\mu^2)$
Pressure	$p_{NH} = 0$	**Boussinesq**	$p_{NH} = h^2\ddot{h}/3$
ε	no assump		no assump.
Type	hyperbolic		dispersive

Lannes, **2013**

Breaking Waves Hyperbolic Model

Axe 1: Model derivation and validation on numerical test cases
– **Wave breaking and dispersion:**
Hyperbolic model with enstrophy descsription
– **Breaking criterion:**
Robust breaking criterion ou no criterion at all

Axe 2 :(Julien Chauchat, LEGI) Sediment transport coupling
Resolution of Exner equation, nonlinear interaction

Validation:
Implementation with TOLOSA project tolosa-project.com
Validation: Delft3D, XBeach

Hydro: experiences in LEGI (*rip currents*) + Mesurements by SHOM

Morpho: From solitary waves on sand beachs, monochromatic and bichromatic waves

State of the art: II Energy dissipation

Advances on wave breaking modelling

Artificial dissipative terms

\+ Viscous term in (Moment Eq)
\+ Convective term in (Mass Eq)

NSWE:
Packwood&Peregrine, **1981**
Boussinesq :
Zelt, **1991**
Wei *et al.*, **1999**

Hybrid method/Switching

− Drop Dispersive terms

Boussinesq type:
Bonneton *et al.*, **2011**
Tissier *et al.*, **2012**
Kazolea *et al.*, **2014**
Duran&Marche, **2015**

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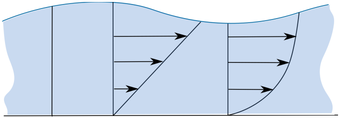
When? Breaking criterion

State of the art: III Turbulent structures generation

Advances on wave breaking modelling

? Assumption on the velocity profile



NO BREAKING

Hyperbolic framework

- Teshukov, **2007** 2D hyperbolic
- Richard&Gavriluyk **2012** Hydraulic jumps
- Ivanova&Gavriluyk **2018** Hydraulic jumps

Dispersive framework

- Castro&Lannes *et al.*, **2014**
- Richard&Gavriluyk *et al.*, **2015** Dispersive
- $\mathcal{MK}$&Noble, **2016** two-layer flow

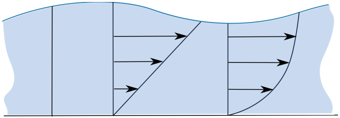
ADD DISSIPATION

- Gavriluyk *et al.*, JFM, **2016** Breaking waves in two-layer model

State of the art: III Turbulent structures generation

Advances on wave breaking modelling

? Assumption on the velocity profile ! Not valid for breaking waves



(flat) (linear) (polynomial)

NO BREAKING

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Breaking waves

Model derivation

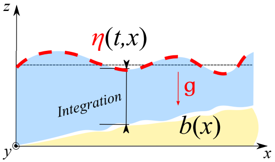
Following

📖 Teshukov **2007**, Richard&Gavrilyuk **2015**

we assume a weakly sheared flow:

$$u(t,x,z) = \mathbf{u}(t,x) + \mu u'(t,x,z),\ \varphi = \frac{\langle\langle u'^2 \rangle\rangle}{h^2}$$

φ is new variable allows to solve turbulence explicitly

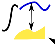


Where the averaged horizontal velocity U is defined as

$$\mathbf{u} = \langle\langle u(t,x,z) \rangle\rangle \equiv \frac{1}{h} \int_{b(x)}^{Z(x,t)} \overline{u}(t,x,z)\ \mathrm{d}z$$

Breaking waves

Model derivation



Navier-Stokes $\longrightarrow$ $\xrightarrow{O(\mu^2)}$ + dissipation

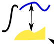
$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(h\boldsymbol{u}) = 0, \\ \dfrac{\partial (h\boldsymbol{u})}{\partial t} + \dfrac{\partial}{\partial x}\left(h\boldsymbol{u}^2 + \dfrac{gh^2}{2} + \dfrac{h^2}{3}\dfrac{D^2 h}{Dt^2} + h^3\varphi\right) = \partial_x\left(h\ \nu_T(x)\ \dfrac{\partial \boldsymbol{u}}{\partial x}\right) + G_b \\ \dfrac{\partial (h\varphi)}{\partial t} + \dfrac{\partial}{\partial x}(h\boldsymbol{u}\varphi) = \nu_T(x)\left(\dfrac{\partial \boldsymbol{u}}{\partial x}\right)^2 - D(x)\ , \quad \dfrac{Dh}{Dt} = \dfrac{\partial h}{\partial t} + \boldsymbol{u}\dfrac{\partial h}{\partial x} \end{cases}$$

Numerical issues:

Dispersive (high-order term) discretization

Breaking waves

Model derivation

Navier-Stokes  $\longrightarrow$ $\xrightarrow{O(\mu^2)}$ + dissipation

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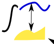

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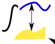
Navier-Stokes  $\longrightarrow$ $\xrightarrow{O(\mu^2)}$ + dissipation

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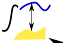
Navier-Stokes  $\longrightarrow$ $\xrightarrow{O(\mu^2)}$ + dissipation

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\end{cases}
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Numerical issues:

Dispersive (high-order term) discretization

Numerical Simulations

Dispersive terms

Change of variables $K = U + \frac{1}{3h}\nabla\left(h^2\dot{h}\right), \alpha = -\frac{2}{3}h^3\left(\frac{\partial U}{\partial x}\right)^2$

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial (hU)}{\partial x} = 0, \\ \dfrac{\partial (hK)}{\partial t} + \dfrac{\partial}{\partial x}\left(hKU + \dfrac{gh^2}{2} + \alpha + h^3\varphi\right) = \dfrac{\partial}{\partial x}\left(\dfrac{4h^3\sqrt{\varphi}}{R}\dfrac{\partial U}{\partial x}\right) - b'hg, \\ \dfrac{\partial h\varphi}{\partial t} + \dfrac{\partial (h\varphi U)}{\partial x} = \dfrac{8h\sqrt{\varphi}}{R}\left(\dfrac{\partial U}{\partial x}\right)^2 - C_r h^3 \varphi^{3/2}, \end{cases}$$

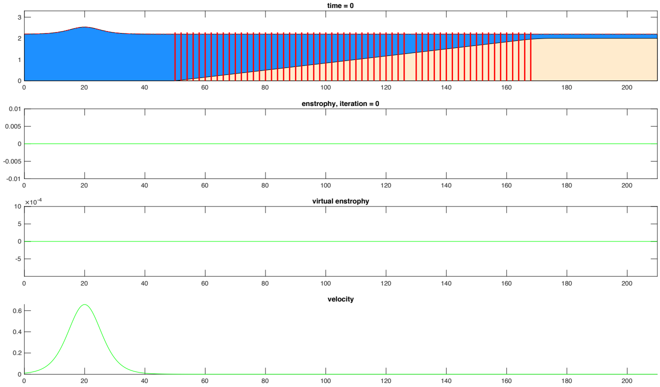
Change of variables $K = U + \frac{1}{3h}\nabla\left(h^2\dot{h}\right), \alpha = -\frac{2}{3}h^3\left(\frac{\partial U}{\partial x}\right)^2$

📖 Le Métayer *et al.*, **2010**

Numerical Simulations

Experimental Data Comparison

Hsiao *et al.*, **2008**, $tg\beta = 0.017$



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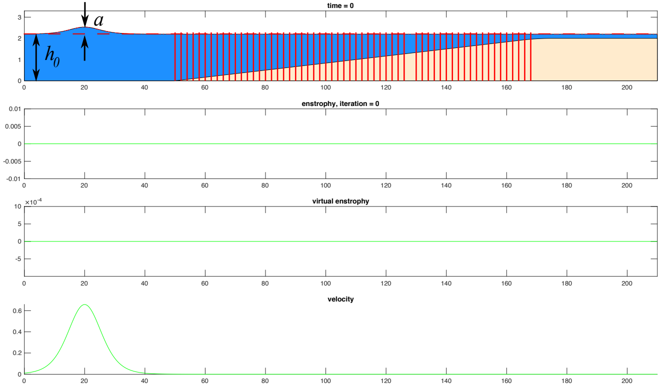

Numerical Simulations

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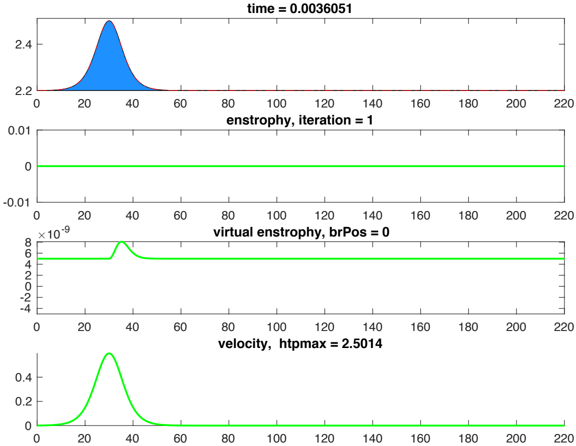
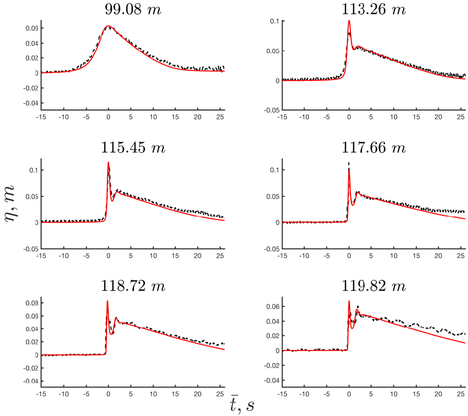


Numerical Simulations

Experimental Data Comparison

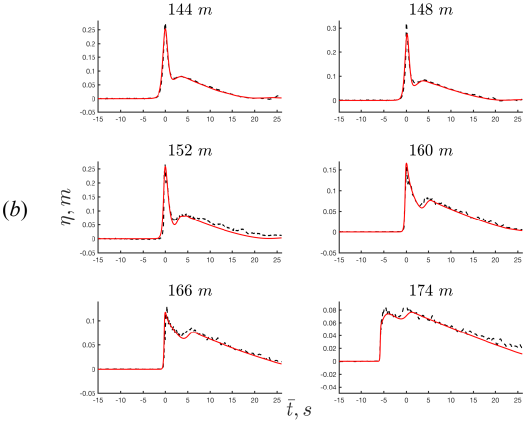
(*a*)



$$N_{trial} = 3,\ h = 1.2m,\ \varepsilon = a_0/h_0 = 0.048$$

Numerical Simulations

Experimental Data Comparison



$$N_{trial} = 41,\ h = 2.2m,\ \varepsilon = a_0/h_0 = 0.137$$

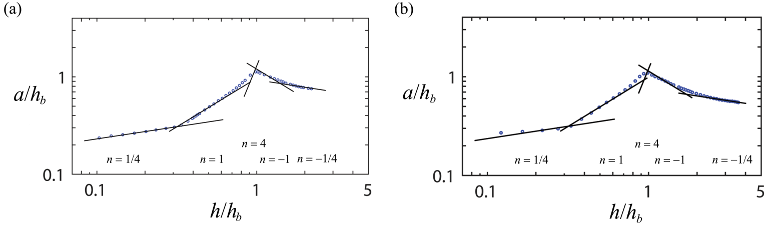
Numerical Simulations

Empirical Laws for breaking depth

$$\frac{\eta_{max}}{h_b} \sim \left(\frac{h_{loc}}{h_b}\right)^n$$

$n = -1/4$ – gradual shoaling, $n = -1$ – rapid shoaling

$n = 4$ – rapid decay, $n = 1$ – zone of gradual decay



Synolakis *et al.*, **1993**

Numerical Simulations

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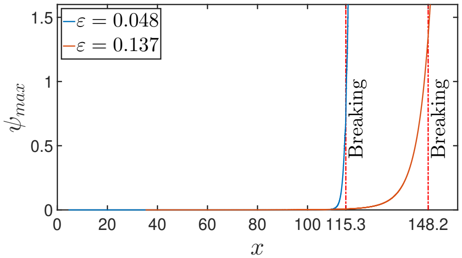
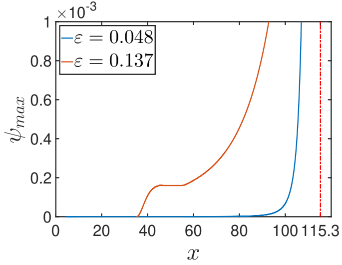
$$\psi_0 = \frac{g}{h_0^*}\widetilde{\psi}_0, \widetilde{\psi}_0 = \begin{cases} \left(0.1 + \dfrac{0.031}{\varepsilon}\right), & \varepsilon > 0.05, \\ 0, & \varepsilon < 0.05, \end{cases} \quad R = \begin{cases} 1.7, & \varepsilon > 0.05, \\ 6, & \varepsilon < 0.05, \end{cases}$$

Entropy evolution

Virtual enstrophy : breaking criterion





$$
\left\{
\begin{array}{l}
\forall t > t_* \quad \text{SGN} + \dfrac{\partial h\psi}{\partial t} + \dfrac{\partial (hU\psi)}{\partial x} = \dfrac{8h\sqrt{\psi}}{R}\left(\dfrac{\partial U}{\partial x}\right)^2 - C_r h^3 \psi^{\frac{3}{2}}, \\
t_* : \max_x(\psi(t_*, x)) \geq \psi_0 \\
\forall t > t_* \;\; \dfrac{\partial h\varphi}{\partial t} + \dfrac{\partial (hU\varphi)}{\partial x} = \dfrac{8h\sqrt{\varphi}}{R}\left(\dfrac{\partial U}{\partial x}\right)^2 - C_r h^3 \varphi^{\frac{3}{2}}.
\end{array}
\right.
$$

Entropy evolution

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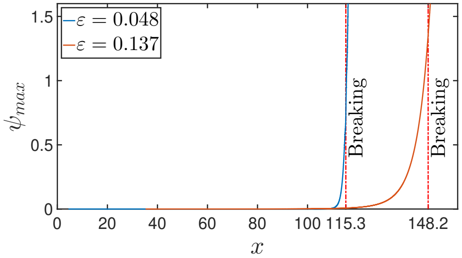

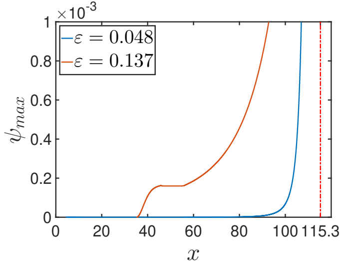

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