Some recent mathematical results on the effective viscosity of suspensions

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Model

A *viscous suspension* is a collection of $N \gg 1$ small rigid particles immersed in a viscous fluid.

Here : spherical particles $B_i = B(x_i, R)$, $1 \leq i \leq N$.

$R \ll 1$ (typical length scale of the flow)

- **Stokes equations** in $\Omega_N := \mathbb{R}^3 \setminus (\cup_{i=1}^N B_i)$:

$$-\Delta u + \nabla p = f, \quad \text{div } u = 0, \quad x \in \Omega_N \tag{St}$$

- **No-slip condition**

$$u|_{\partial B_i} = v_i + \omega_i \times (x - x_i), \quad \forall i. \tag{NS}$$

- Newton's dynamics

$$
\begin{aligned}
\dot{x}_i &= v_i, \\
m\dot{v}_i &= \int_{\partial B_i} \sigma(u,p)n + f_i, \\
J\dot{\omega}_i &= \int_{\partial B_i} \sigma(u,p)n \times (x - x_i) + t_i
\end{aligned}
\tag{N}
$$

$\sigma(u,p) = 2D(u) - p\mathrm{Id}$ Newtonian stress tensor.

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Later on, we will neglect inertia:

$$\begin{aligned} \dot{x}_i &= v_i, \\ 0 &= \int_{\partial B_i} \sigma(u,p)n + f_i \\ 0 &= \int_{\partial B_i} \sigma(u,p)n \times (x - x_i) + t_i \end{aligned} \qquad \text{(N')}$$

No contact : $(x_i) \in \mathcal{U} = \{X \in \mathbb{R}^{3N}, \quad |x_i - x_j| > 2R, i \neq j\}$.

Remark: For any $f \in L^{6/5}(\mathbb{R}^3)$, $X = (x_i) \in \mathcal{U}$, $V = (v_i)$, $\omega = (\omega_i)$, there is a unique solution of (St)-(NS), linear in (V, ω):

$$u = u[X, V, \omega](x) \in \dot{H}^1(\Omega_N)$$

Back to (N): yields an ODE

$$\begin{pmatrix} \dot{X} \\ v \\ \omega \end{pmatrix} = \mathcal{F}_N \begin{pmatrix} X \\ v \\ \omega \end{pmatrix}$$

Back to (N'): balance of forces and torques yields an invertible linear system on $\binom{V}{\omega}$, with coefficients depending on X:

$$\dot{X} = \mathcal{F}_N(X)$$

Theorem [Hillairet-Sabbagh'22]

For any initial data with $X^{init} \in \mathcal{U}$, these ODES are well-posed globally in time, with $X(t) \in \mathcal{U}$ for all times.

Large N asymptotics

General Questions :

- Behaviour of $u = u_N$, $X = (x_{i,N})$, $V = (v_{i,N})$, $\omega = \omega_{i,N}$ as $N \to +\infty$?
- Derivation of a reduced (continuous) model ?

Here :

- what is the mean effect of the rigid particles on the viscosity of the suspension ?
- Can we derive a fluid model, with an effective viscosity ?

Subproblem: PDE block

(St)+(NS) + balance of forces and torques (and $f_i = t_i = 0$).

Asymptotics of this subsystem, under geometrical and statistical assumptions on the distribution of $x_1, \dots, x_N$?

Examples :

- Convergence of the empirical measure:

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i} \to \rho = \rho(x)$$

  ρ bounded, supported in the closure of a bounded domain $\mathcal{O}$.
  Homogeneous case : $\rho = \frac{1}{|\mathcal{O}|}1_{\mathcal{O}}$.

- Assumption on the minimal distance between the particles.

- Stationarity :

  Given $0 < \varepsilon \ll 1$, a bounded domain $\mathcal{O}$, and a stationary point process $\mathcal{X}$ on $\mathbb{R}^3$ of intensity 1. Then,

$$\{x_1, \dots, x_N\} = \varepsilon\mathcal{X} \cap \mathcal{O}$$

- Addition of mixing assumptions . . .

Dilute suspensions

Solid volume fraction:

$$\phi := N\frac{4}{3}\pi R^3/|\mathcal{O}| \quad \text{small}$$

Two different types of dilutions:

i) Large interparticle distance. Typically

$$|x_i - x_j| \geq cN^{-1/3}$$

ii) Thinning of a point process: [GV'21, Duerinckx-Gloria'21]

Case **i) The method of reflections** :

Approximation through an iteration using single ball solutions.

$$u^{app} = u^{\emptyset} + \sum_{i=1}^{N} u_i^1 + \sum_{i=1}^{N} u_i^2, + \dots, \qquad \text{the same for } v_i^{app}, \omega_i^{app}.$$

- $u^{\emptyset}$ sees no ball:

$$-\Delta u^{\emptyset} + \nabla p^{\emptyset} = f, \quad \text{div } u^{\emptyset} = 0 \quad \text{in } \mathbb{R}^3.$$

- u_i^1 corrects $u^{\emptyset}$ on B_i, neglecting B_j, $j \neq i$.

$$\begin{aligned} -\Delta u_i^1 + \nabla p_i^1 &= 0, \quad \text{div } u_i^1 = 0 \quad \text{in } \mathbb{R}^3 \setminus B_i \\ u_i^1|_{\partial B_i} &= v_i^1 + \omega_i^1 \times (x - x_i) \\ &\quad - u^{\emptyset}(x_i) - \omega^{\emptyset}(x_i) \times (x - x_i) - D(u^{\emptyset})(x_i)(x - x_i) \\ \int_{\partial B_i} \sigma(u_i^1, p_i^1) n &= 0, \quad \dots \end{aligned}$$

- u_i^2 corrects u_j^1, $j \neq i$, neglecting B_j, $j \neq i$, *etc*

Explicit solutions at each step.

Effective viscosity : One finds a *stresslet*

$$u_i = u[D(u^\emptyset)(x_i)](x - x_i) \quad \text{on } \mathbb{R}^3 \setminus B_i$$

$$u[S](x) = -\frac{3S : x \otimes x}{8\pi |x|^5} x - 3\frac{R^2}{20}\frac{Sx}{|x|^5} + 3R^2 \frac{S : (x \otimes x)}{|x|^7} x.$$

On $\mathbb{R}^3$, after extension:

$$-\Delta u^{app} + \nabla p^{app} = f 1_{\Omega_N} + \text{div}\left(\frac{5\phi}{N}|\mathcal{O}| \sum_i D(u^\emptyset)(x_i) \frac{\sigma_{\partial B x_i}}{4\pi R^2}\right) + O(R^2)$$

If $N \to +\infty$, at order $O(\phi)$:

$$-\Delta u^{eff} + \nabla p^{eff} = (1 - \phi) f + 5\phi |\mathcal{O}| \text{div}\left(D(u^\emptyset)\rho\right), \ \text{div } u^{eff} = 0.$$

The first equation can be replaced consistently at order ϕ by:

$$-\text{div}\left(2\left(1 + \frac{5}{2}\phi|\mathcal{O}|\rho\right) D(u^{eff})\right) + \nabla p^{eff} = (1 - \phi) f$$

Remark : the effective model is a Stokes equation with a modified viscosity coefficient $\mu^{eff} \neq 1$. In the homogeneous case,

$$\mu_{eff} = (1 + \frac{5}{2}\phi) \quad \text{in } \mathcal{O} \quad : \quad \text{Einstein's viscosity}$$

Difficulties: justifying the method of reflections may be hard, because stresslets have

1) a slow decay : obstacle to the convergence of $\sum_{i=1}^{N} u_i^k$.
2) a singularity at the origin : when particles are close, error terms are not so small.

Refs: [Sanchez-Palencia'82],[Haines-Mazzucato'10], [Niethammer-Schubert'19],[Hillairet-Wu'19], [GV-Höfer'20], [Duerinckx-Gloria'21]

Einstein's formula is now proved for ϕ small independent of N, under minimal conditions.

Example: [GV-Höfer'20]

$$\exists \delta > 0, \quad \forall i \neq j, \; |x_i - x_j| \geq (2+\delta)R \qquad \text{(A2')}$$

$$\exists C, \alpha > 0, s.t. \quad \forall \eta, \quad \sharp\{i, |x_i - x_j| \leq \eta N^{-1/3}\} \leq C\eta^{\alpha} N. \quad \text{(A2'')}$$

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Question: Can we go beyond Einstein's formula ? Up to $o(\phi^2)$?

Various formulas in the literature, for periodic and random stationary distributions of particles ... formulas do not always coincide !

Difficulties:

- Pairwise interactions must be taken into account.

- Microscopic structure plays a role: **knowing ρ is not enough**.

One can show in the homogeneous case that if μ_2 exists, it is given by:

$$\mu_2 S : S = \lim_N \left(\frac{1}{N^2} \sum_{i \neq j} \mathcal{M}(x_i - x_j) - \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mathcal{M}(x-y)\rho(x)\rho(y)dxdy \right)$$

for $\mathcal{M}$ a Calderon-Zygmund operator (degree -3).

Remark: the limit is not zero ! Due to the singularity.

Question: Under what conditions on the x_i's does this mean field limit exist ?

OK under stationarity and separation assumptions: combines

- arguments à la Serfaty in the analysis of Coulomb gases
- homogenization arguments.

Connection to homogenization

We focus on the Stokes subproblem (in a bounded domain Ω):

$$-\Delta u + \nabla p = f, \quad \text{div } u = 0 \in \Omega_N,$$

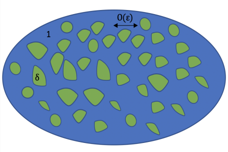
$$u = u_i + \omega_i \times (x - x_i) \quad \text{at } \partial B_i, \quad u|_{\partial\Omega} = 0$$

$$\int_{\partial B_i} \sigma(u,p)n = \int_{\partial B_i} \sigma(u,p)n \times (x - x_i) = 0.$$

Question : What if $\phi \sim 1$? No perturbative approach.

"Degenerate problem of homogenization": obtained as the limit when $\mu \to \infty$ of a Stokes problem with viscosity coefficient

$\mu_N = 1$ in Ω_N, $\quad \mu_N = \mu$ in $\cup B_i$



Analogue problem for the laplacian: studied in depth by Jikov.

Extension to Stokes by [Duerinckx-Gloria]: under usual stationarity and ergodicity conditions, and if

$$|x_i - x_j| \geq (2+\delta)\, R, \quad \delta > 0, \quad \forall i \neq j$$

the solution $u = u_N$ converges as $N \to +\infty$ to the system

$$\begin{aligned} -2div(\overline{A}D(u)) + \nabla p = (1-\phi)f, \quad & \text{in } \Omega, \\ \text{div } u = 0, \quad & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 & \end{aligned}$$

Remark : One can relax much the assumption on the minimal distance: still true under a moment bound on the diameter of the clusters of close particles.

Network approximation for dense suspensions

Introduced by [Borcea'98], [Berlyand'01], ... See [Berlyand-Kolpakov-Novikov'13], [GV-Girodroux-Lavigne'22]. .

A tool to treat dense suspensions. So far, mostly used for fixed N.

Crucial observation: If two balls B_i and B_j of unit radius are δ_{ij} close, the energy of the solution u of

$$-\Delta u + \nabla p = 0, \quad \text{div } u = 0, \qquad \text{in } \mathbb{R}^3 \setminus (B_i \cup B_j)$$
$$u|_{\partial B_i} = v_i, \quad u|_{\partial B_j} = v_j,$$

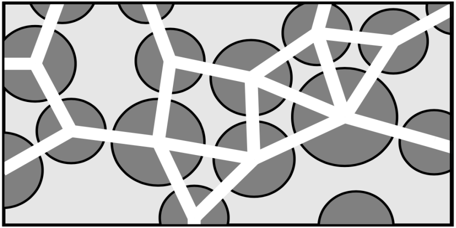
has an energy that scales like

$$\int |\nabla u|^2 \sim \frac{|v_i - v_j|^2}{\delta_{ij}}$$

Model reduction:

One can replace the continuous geometry by a weighted graph G:

$$\begin{cases} i \sim j \quad \text{if } B_i, B_j \text{ neighbours} \\ \text{weight } \delta_{ij} \end{cases}$$



Finiteness of the energy of the corrector corresponds to boundedness of discrete minimal energies on G_L (restriction to balls in the cube of size L), of the form

$$\mathcal{E}(G_L, S) = \min_{(v_i)} \frac{1}{L^3} \sum_{i \sim j} \frac{|v_i - v_j|^2}{\delta_{ij}} + \frac{1}{L^3} \sum_i |u_i - Sx_i|^2$$

In particular, these energies are bounded under a moment condition on the clusters.

Open problems:

- better understanding of the asymptotics of these discrete energies.
- what is the limit of u_N when the corrector is not well-defined ?
- coupling with the particles ?