The hydrostatic approximation for stratified fluids

An open problem

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Outline

1. Context
2. The problem with hydrostatic equations
3. A result with eddy diffusivity
4. Benefits from isopycnal coordinates

The initial set of equations

Motivated by geophysical flows, we study the heterogeneous, incompressible Euler equations with gravity force:[1]

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) + \partial_z(\rho w) = 0, \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + w\partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \rho\big(\partial_t w + \mathbf{u} \cdot \nabla_{\mathbf{x}} w + w \partial_z w\big) + \partial_z P + g\,\rho = 0, \\ \nabla_{\mathbf{x}} \cdot \mathbf{u} + \partial_z w = 0, \\ \text{boundary conditions.} \end{cases} \tag{E}$$

The velocity field is denoted $(\mathbf{u}, w) : \Omega \to \mathbb{R}^d \times \mathbb{R}$, the density $\rho : \Omega \to \mathbb{R}_\star^+$.
The pressure $P : \Omega \to \mathbb{R}$ is reconstructed through the elliptic equations

$$-\nabla_{\mathbf{x}} \cdot \big(\tfrac{1}{\rho} \nabla_{\mathbf{x}} P\big) - \partial_z\big(\tfrac{1}{\rho}\partial_z P\big) = \nabla_{\mathbf{x}} \cdot \big((\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + w\partial_z \mathbf{u}\big) + \partial_z\big(g\rho + \mathbf{u} \cdot \nabla_{\mathbf{x}} w + w\partial_z w\big).$$

Using the hydrostatic approximation, the system becomes ...

[1] We are discarding the Coriolis force because we are interested in short-time stability. We are also discarding (for now) eddy viscosity and diffusivity.

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The hydrostatic approximation

The heterogeneous, incompressible Euler equations with gravity force

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with the hydrostatic approximation becomes

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Qn: stability of shear flows $(\rho, \mathbf{u})(t, \mathbf{x}, z) = (\underline{\rho}(z), \underline{\mathbf{u}}(z))$?

(partial and biased) state of the art on the hydrostatic equations

Homogeneous case: $\rho \equiv 1$.

- Spectral stability of the linearized system about shear flows, under the Rayleigh criterion $\underline{\mathbf{u}}''(z) \neq 0$. [Rayleigh (1880)][Arnold][Drazin&Reid]
- Ill-posedness of the linearized system about certain shear flows. [Renardy '09]
- Well-posedness of the (nonlinear) system in Sobolev spaces under the Rayleigh criterion. [Masmoudi&Wong '12] (after [Grenier '99], [Brenier '03])

Stably stratified case: $\partial_z \rho < 0$.

- Spectral stability of the linearized system about shear flows, under the Miles and Howard criterion $\frac{1}{4}|\underline{\mathbf{u}}'(z)|^2 \leq \frac{-\underline{\rho}'(z)}{\underline{\rho}(z)}$. [Miles '61][Howard '61]
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Open problem

Well-posedness of the initial-value problem for the (linear or nonlinear) system?

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The linearized system about shear flows

Let us linearize the system

$$\begin{cases} \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) + \partial_z(\rho w) = 0, \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} + w\partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \partial_z P = -\rho, \\ \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \\ \text{boundary conditions (periodic).} \end{cases} \tag{H}$$

about shear solutions $(\rho, \mathbf{u})(t, \mathbf{x}, z) = (\underline{\rho}(z), \underline{\mathbf{u}}(z))$. We get

$$\begin{cases} \partial_t \rho + \underline{\mathbf{u}}(z) \cdot \nabla_{\mathbf{x}} \rho + \underline{\rho}'(z) w = 0, \\ \partial_t \mathbf{u} + (\underline{\mathbf{u}}(z)\cdot\nabla_{\mathbf{x}})\mathbf{u} + \underline{\mathbf{u}}'(z) w + \frac{1}{\underline{\rho}(z)} \nabla_{\mathbf{x}} P = 0, \\ \partial_z P = -\rho, \qquad\qquad (\Longrightarrow P = \int_\cdot^1 \rho \, \mathrm{d}z \overset{\text{def}}{=} \mathrm{L}\rho) \\ \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \qquad (\Longrightarrow w = -\int_0^\cdot \nabla_{\mathbf{x}} \cdot \mathbf{u} \, \mathrm{d}z = -\mathrm{L}^\star \nabla_{\mathbf{x}} \cdot \mathbf{u}) \\ \text{boundary conditions (periodic).} \end{cases} \tag{L}$$

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about shear solutions $(\rho, \mathbf{u})(t, \mathbf{x}, z) = (\underline{\rho}(z), \underline{\mathbf{u}}(z))$. We get

$$\begin{cases} \partial_t \rho + \underline{\mathbf{u}}(z) \cdot \nabla_{\mathbf{x}} \rho - \underline{\rho}'(z) \mathrm{L}^\star \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\underline{\mathbf{u}}(z) \cdot \nabla_{\mathbf{x}})\mathbf{u} - \underline{\mathbf{u}}'(z) \mathrm{L}^\star \nabla_{\mathbf{x}} \cdot \mathbf{u} + \frac{1}{\underline{\rho}(z)} \nabla_{\mathbf{x}} \mathrm{L} \rho = 0, \\ \text{boundary conditions (periodic).} \end{cases} \tag{L}$$

The linearized system enjoys a symmetric structure if $\underline{\rho}'(z) < 0$, $\underline{\mathbf{u}}'(z) = \mathbf{0}$.

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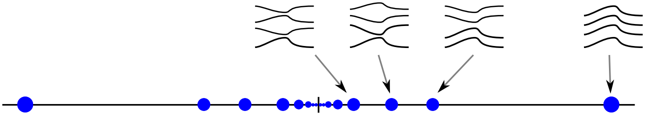

The linearized system about shear flows

$$\begin{cases} \partial_t \rho + \underline{\mathbf{u}}(z)\cdot\nabla_{\mathbf{x}}\rho - \underline{\rho}'(z)\mathrm{L}^\star\nabla_{\mathbf{x}}\cdot\mathbf{u} = 0, \\ \partial_t\mathbf{u} + (\underline{\mathbf{u}}(z)\cdot\nabla_{\mathbf{x}})\mathbf{u} - \underline{\mathbf{u}}'(z)\mathrm{L}^\star\nabla_{\mathbf{x}}\cdot\mathbf{u} + \frac{1}{\underline{\rho}(z)}\nabla_{\mathbf{x}}\mathrm{L}\rho = 0, \\ \text{boundary conditions (periodic).} \end{cases} \tag{L}$$

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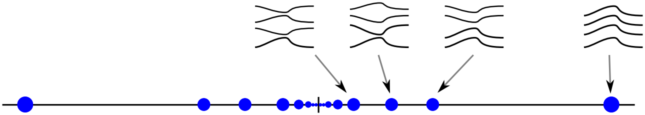

The symmetric structure is ruined if $\underline{\mathbf{u}}'(z) \neq \mathbf{0}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big(\,\rho\,,\ \tfrac{-1}{\underline{\rho}'(z)}\rho\,\big)_{L^2_{\mathbf{x},z}} + \big(\,\mathbf{u}\,,\ \underline{\rho}(z)\mathbf{u}\,\big)_{L^2_{\mathbf{x},z}}\Big) = \big(\,\mathbf{u}\,,\ \underline{\rho}(z)\underline{\mathbf{u}}'(z)\mathrm{L}^\star\nabla_{\mathbf{x}}\cdot\mathbf{u}\,\big)_{L^2_{\mathbf{x},z}}.$$

$\rightsquigarrow$ no (obvious) control of an energy.

Conclusion

$$\begin{cases} \partial_t \rho + \underline{\mathbf{u}}(z) \cdot \nabla_{\mathbf{x}} \rho - \underline{\rho}'(z) \mathrm{L}^\star \nabla_{\mathbf{x}} \cdot \mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\underline{\mathbf{u}}(z) \cdot \nabla_{\mathbf{x}}) \mathbf{u} - \underline{\mathbf{u}}'(z) \mathrm{L}^\star \nabla_{\mathbf{x}} \cdot \mathbf{u} + \frac{1}{\underline{\rho}(z)} \nabla_{\mathbf{x}} \mathrm{L} \rho = 0, \\ \text{boundary conditions (periodic).} \end{cases} \tag{L}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big(\big(\rho \,,\, \tfrac{-1}{\underline{\rho}'(z)} \rho \big)_{L^2_{\mathbf{x},z}} + \big(\mathbf{u} \,,\, \underline{\rho}(z) \mathbf{u} \big)_{L^2_{\mathbf{x},z}} \Big) = \big(\mathbf{u} \,,\, \underline{\rho}(z) \underline{\mathbf{u}}'(z) \mathrm{L}^\star \nabla_{\mathbf{x}} \cdot \mathbf{u} \big)_{L^2_{\mathbf{x},z}}.$$

Possible high-frequency instabilities
in the hydrostatic framework
in the presence of shear velocity, $\underline{\mathbf{u}}'(z) \neq 0$,
even under the Miles & Howard criterion.
The stable stratification, $\underline{\rho}'(z) < 0$, helps (a bit).

Possible ways around:

- Work in the analytic framework. e.g. [Kukavica, Temam, Vicol & Ziane '11]
- Add (horizontal) viscosity. e.g. [Cao, Li & Titi '16]
- Relax the hydrostatic approximation [Desjardins, Lannes & Saut '21]

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Eddy viscosity and diffusivity

The system studied in [Cao, Li & Titi '16] (among other works) is

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla_{\mathbf{x}} \cdot \rho + w \partial_z \rho = \kappa \Delta_{\mathbf{x}} \rho, \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} + w\partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = \nu \Delta_{\mathbf{x}} \mathbf{u}, \\ \partial_z P = -\rho, \quad \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \\ \text{boundary conditions (periodic).} \end{cases} \tag{$\mathrm{H}_{\nu,\kappa}$}$$

The horizontal eddy viscosity ($\nu > 0$) and diffusivity ($\kappa > 0$) approximate effective isopycnal viscosity and diffusivity:
[Gent & McWilliams '90] propose the following parameterization

$$\begin{cases} \partial_t \rho + (\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}} \rho + (w + w_\star)\partial_z \rho = 0, \\ \rho\big(\partial_t \mathbf{u} + ((\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_{\mathbf{x}})\mathbf{u} + (w + w_\star)\partial_z \mathbf{u}\big) + \nabla_{\mathbf{x}} P = 0, \\ \partial_z P = -\rho, \quad \partial_z w = -\nabla_{\mathbf{x}} \cdot \mathbf{u}, \\ \text{boundary conditions.} \end{cases} \tag{H_{κ}}$$

with
$$\mathbf{u}_\star = \kappa \partial_z \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right) , \quad w_\star = -\kappa \nabla_{\mathbf{x}} \cdot \left(\frac{\nabla_{\mathbf{x}} \rho}{\partial_z \rho} \right) .$$

Main result

$$\begin{cases} \partial_t \rho + (\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_\mathbf{x} \rho + (w + w_\star)\partial_z \rho = 0, \\ \rho\big(\partial_t \mathbf{u} + ((\mathbf{u} + \mathbf{u}_\star) \cdot \nabla_\mathbf{x})\mathbf{u} + (w + w_\star)\partial_z \mathbf{u}\big) + \nabla_\mathbf{x} P = 0, \\ \partial_z P = -\rho, \quad \partial_z w = -\nabla_\mathbf{x} \cdot \mathbf{u}, \\ \text{boundary conditions (free surface)}, \\ \mathbf{u}_\star = \kappa \partial_z \left(\frac{\nabla_\mathbf{x} \rho}{\partial_z \rho}\right) , \quad w_\star = -\kappa \nabla_\mathbf{x} \cdot \left(\frac{\nabla_\mathbf{x} \rho}{\partial_z \rho}\right) . \end{cases} \qquad (\mathrm{H}_\kappa)$$

[R. Bianchini & VD]

For sufficiently regular data satisfying the (strict) stable stratification assumption

$$-\partial_z \rho|_{t=0} \geq \alpha > 0$$

and for any $\kappa > 0$, there exists a unique (classical) solution to (H_κ) on the time interval $[0, T]$ with

$$T^{-1} = C\left(1 + \kappa^{-1}\big(\big|\underline{\mathbf{u}}'\big|_{L^2_z}^2 + M_0^2\big)\right),$$

where M_0 is the size of the initial deviation from the shear flow equilibrium $(\underline{\rho}(z), \underline{\mathbf{u}}(z))$, and C depends only on M_0, α and the size of $(\underline{\rho}(z), \underline{\mathbf{u}}(z))$.

Main result

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Remarks.

- We do not use viscosity (only diffusivity), nor analytic data;
- The destabilizing role of the shear velocity is apparent;
- Stable stratification is used in a crucial way in our proof;

$\rightsquigarrow$ we make use of isopycnal coordinates.

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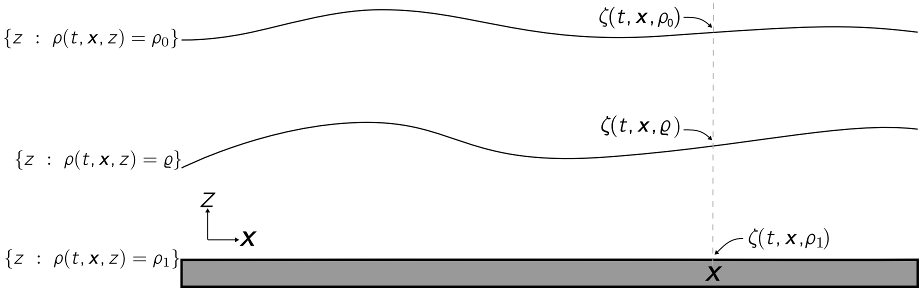
Isopycnal coordinates

We define the variable $h(t,\mathbf{x},r)>0$ from the density $\rho(t,\mathbf{x},z)$ through

$$h \stackrel{\text{def}}{=} -\partial_r\zeta, \qquad \rho(t,\mathbf{x},\zeta(t,\mathbf{x},r)) = r, \quad \zeta(t,\mathbf{x},\rho(t,\mathbf{x},z)) = z.$$



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The system

$$\begin{cases} \partial_t\rho + (\mathbf{u}+\mathbf{u}_\star)\cdot\nabla_\mathbf{x}\rho + (w+w_\star)\partial_z\rho = 0, \\ \rho\big(\partial_t\mathbf{u} + ((\mathbf{u}+\mathbf{u}_\star)\cdot\nabla_\mathbf{x})\mathbf{u} + (w+w_\star)\partial_z\mathbf{u}\big) + \nabla_\mathbf{x}P = 0, \\ \partial_z P = -\rho, \quad \partial_z w = -\nabla_\mathbf{x}\cdot\mathbf{u}, \\ \text{boundary conditions (free surface).} \end{cases} \tag{H_κ}$$

reads in <u>isopycnal coordinates</u>

$$\begin{cases} \partial_t h + \nabla_\mathbf{x}\cdot(h\mathbf{u}) = \kappa\Delta_\mathbf{x} h, \\ r\Big(\partial_t\mathbf{u} + \big((\mathbf{u} + \kappa\frac{-\nabla_\mathbf{x} h}{h})\cdot\nabla_\mathbf{x}\big)\mathbf{u}\Big) + \nabla_\mathbf{x}\psi = 0, \end{cases} \tag{H_κ}$$

where

$$\psi(t,\mathbf{x},r) \stackrel{\text{def}}{=} \rho_0\int_{\rho_0}^{\rho_1} h(t,\mathbf{x},r')\,\mathrm{d}r' + \int_{\rho_0}^{r}\int_{r'}^{\rho_1} h(t,\mathbf{x},r'')\,\mathrm{d}r''\,\mathrm{d}r'.$$

The system in isopycnal coordinates

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Remarks:

- The eddy diffusivity of [Gent & McWilliams '90] is nice and simple.
- The advection in the variable z (or r) has disappeared.
- The domain is flattened.
- The system is easily discretized (multilayer framework) [Adim, w.i.p.]
- The system enjoys a partial symmetric structure analogous as the one exhibited in the Eulerian coordinates, but the symmetry defect involves an extra derivative on $\zeta \stackrel{\text{def}}{=} \int_r h$ (and not $\mathbf{u}$).

$\rightsquigarrow$ The proof of our main result is based on the energy method on (H_κ), using product, commutator, composition estimates in anisotropic Sobolev spaces.

Thoughts to go

- The well-posedness of the heterogeneous hydrostatic equations, without diffusivity or viscosity, is an open problem.
- Isopycnal coordinates are interesting for numerical <u>and also</u> theoretical analyses.

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Thank you for your attention